Integrating Mathematics and History: The Scholarship of D D Kosambi

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Today, D D Kosambi's significance as a historian greatly overshadows his reputation and contributions in mathematics. Kosambi simultaneously worked in both areas for much of his adult life, and to understand the body of his work either in the social sciences or in mathematics, an appreciation of the complementarity of his interests is essential. An understanding of Kosambi the historian can only be enhanced by an appreciation of Kosambi the mathematician. In a fundamental way, Kosambi embodied the multidisciplinary approach, channelling diverse interests – indeed combining them – to create scholarship of high order.

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D D Kosambi [1] is widely regarded as the scholar who revolutionised the study of Indian history by bringing diverse academic strands that included mathematics, statistics, Marxist thought and critical analysis to historiography. He also made important contributions in the areas of genetics, numismatics, Indology, literary criticism and Sanskrit studies, in addition to his considerable work in mathematics, which he taught all his working life. He thus occupies a very special place in the intellectual development of modern India.

Today, Kosambi's significance and his contribution as a historian greatly overshadows his reputation and contributions in mathematics. He simultaneously worked in both areas for much of his adult life, and to understand the body of his work either in the social sciences or in mathematics, an appreciation of the complementarity of his interests is essential. An understanding of Kosambi the historian can only be enhanced by an appreciation of Kosambi the mathematician. For those not trained in the subject, the mathematics may itself be subtle and difficult to follow in its entirety, but a knowledge of the intellectual preoccupations, the questions that concerned him, and the techniques and tools at his disposal can help in bringing out the very natural manner in which DDK's mathematics informed and refined his approach to history. In a fundamental way, Kosambi embodied the multidisciplinary approach, channelling diverse interests – indeed combining them – to create scholarship of high order [2].

The constancy of Kosambi's interest and work in mathematics is telling, and a significant component of how and why he was so influential as a historian. From 1930, when his first contribution [3] appeared in the *Indian Journal of Physics*, to 1966, when his article in *Scientific American* entitled "Numismatics as a Science" [4] appeared shortly before he died, DDK authored about 65 articles which are either on mathematics, statistics, or use either of these fields as integral components of his studies in other areas [5].

Kosambi's career as a mathematician might have taken a very different course if he had managed to stay on at Harvard for a PhD, possibly with William E Graustein [6] or with George D Birkhoff [7], both early mentors. Owing to a combination of reasons, though, the *Phi Beta Kappa* scholar who had graduated cum laude returned to India to take up a teaching position at Banaras Hindu University in 1929, without the doctorate. He moved to Aligarh Muslim University in 1931, to join a department of mathematics that included the French mathematician, Andre Weil, and a number of colleagues of note such as T K Vijayaraghavan. He also started publishing independently in these years, articles of both a pedagogic nature as well as original contributions.

Early Work

One early work that stands out, a testimony to a sense of mischief and humour and to an age, is his paper "On a Generalisation of the Second Theorem of Bourbaki" which was "a parodic note passed off as a serious contribution to a provincial journal" [8], the *Bulletin of the Academy of Sciences*, UP [9]. Bourbaki was a fiction, the creation (and eventually the nom-de-plume) of a group of avant-garde French mathematicians that included Weil. When DDK was finding it difficult to deal with a senior colleague at Aligarh, Weil apparently suggested to him that he use the Bourbaki myth to deflate the senior's ego. Kosambi was more than happy to comply, and in the process, the name of Bourbaki first entered the scientific literature in the title of this paper [10]. Kosambi got completely into the spirit of things, discarding not one but three *Vysokoblagodaren axioms* in the process and killing off a D Bourbaki through lead poisoning during the Revolution [9]. The French group went on to systematise much of modern mathematics through several books and monographs that appeared subsequently as from the "Bourbaki school" [10].

Nearly 10 years were to pass before Kosambi wrote his first article that dealt with a non-mathematical subject. In 1939, "A Note on the Trial of Socrates" appeared in the magazine of Fergusson College, Pune, to which DDK had moved by then. He had also acquired a respectable curriculum vitae with over 20 publications, many of them on "path-geometry" [11], an area of study that was to occupy him for several decades. He had made valuable contacts with some leading geometers in Europe, among them being T Levi-Civita and E Cartan who communicated his papers [12] and commented on them. His linguistic talents were also in evidence – in these 10 years, he had published articles in French and German, and had even had an article translated into Japanese. In later years, he was even to write an article on the method of least–squares that was first published in Chinese, and only later in English [13].

An article in a college magazine is one thing, a proper scholarly article in the social sciences is quite another. Kosambi's initial papers outside mathematics were to appear in 1940, in the *Annals of the Bhandarkar Oriental Research Institute*, titled "The Emergence of National Characteristics among Three Indo-European People" [14] and in the *New Indian Antiquary*, "A Note on Two Hoards of Punch-Marked Coins Found at Taxila" [15]. The latter was a critique and review of earlier work, and was preceded and succeeded by a number of articles in the Bangalore journal, *Current Science*, on ancient Indian coins, their composition, and their weights [16, 17].

As has been extensively documented, the novelty of Kosambi's approach to numismatics lay in his use of statistical techniques. His research was intensive. Starting with the paper "A Statistical Study of the Weights of Old Indian Punch-Marked Coins" in *Current Science* [17], he carried out a series of systematic studies, papers such as "On the Study and Metrology of Silver Punch-Marked Coins, A Bivariate Extension of Fisher's z-test" and "Correlation and Time-Series" and "On the Origin and Development of Silver Coinage in India". He was also well aware that this analysis had a wider applicability and could have significant consequences in other areas of inquiry.

From this point onwards, although he published in both areas, Kosambi's publications in the social sciences far exceeded those in mathematics, and as the impact of his work in Indian history increased, as a mathematician his development did not proceed at a comparable rate.

Main Mathematical Contributions

For the record, in all DDK authored around 150 publications, including articles, essays, monographs, and books. Of the 65 or so that were in mathematics and statistics, there are both articles of a pedagogical nature as well as original papers. The principal mathematical contributions that he is known for stem from two papers that he wrote in quick succession, one that appeared in the *Journal of the Indian Mathematical Society*, and one in the *Annals of Eugenics* in 1943 and 1944. That this was done when he was a college lecturer, at a time when the country was at war, and when the national movement was at fever pitch is all the more significant.

In perspective, the 1940s were the best time for Kosambi's creative genius. In the period from his mid-30s to his mid-40s, he seemed to have found his métier, the synthesis of ideas in diverse areas. While still in Pune, where he had limited access to libraries or critical colleagues, he branched out into new areas of enquiry, created new disciplines, came up with revolutionary methods of analysis, and essentially developed into the polymath that he is known to have been. When he moved to the Tata Institute of Fundamental Research (TIFR), he continued to live in Pune for some of the time, famously commuting to Bombay by train on the Deccan Queen several times a week, getting a considerable amount of his work done in transit, but spending little real time in TIFR. Although he would be on the rolls there for something like 15 years, this tenure would end on a bitter note.

In addition to the two papers mentioned above, one other work of DDK's done around the same time illustrates the syncretic nature of his approach to scholarship. A little background may be appropriate, given that the basic question remains interesting and relevant even today. The Harvard philologist, George Kingsley Zipf had, in the course of his analysis of a variety of texts, been led to propound a law that pertains to the frequency of word usage in natural language texts. Zipf's law states that the number of times a given word is used in a natural language is inversely proportional to its rank. Thus, the most frequently used word in the English language, "the" is used about twice as often as the second most frequently used word "of", and three times as frequently as the word of rank three, namely, "and" and so on. A number of languages are known to obey this "law", and indeed, papers continue to appear on the unexpected occurrence and validity of Zipf's law in a number of different contexts, notably city sizes, and DNA sequences [18].

Kosambi was not in consonance with a related observation of Zipf's on word frequencies, and he elaborated this disagreement in the paper "On Valid Tests of Linguistic Hypotheses" [19] that also appeared during this time in the *New Indian Antiquary*.

This work illustrates the intellectual synthesis, the breadth of knowledge – and acerbity! – that DDK possessed. Kosambi felt that Zipf's work was poorly validated, and his attack was direct [19], starting

> It is known that in any connected piece of writing ('language stream') the number of words used twice is far less than that used only once. The number occurring three times is still less and the drop continues rapidly. [...] Zipf has proposed a 'law' for this, the number of words used n times being, according to him, proportional to n^{-2}. The main purpose of this note is to raise serious objections to this inverse square 'law'. These objections are statistical. I maintain that no such law, whatever the exponent, will do for the data so given because the fit is not sufficiently good even when the best exponent is taken by calculations on the logarithmic scale. To put this in non-technical language: to every head, there will be the cube-shaped wooden box that fits best, but in general, a rubber cap or a hat of the right size will fit better, and the latter is more likely to indicate a contour of the skull.

DDK then went on to present a statistical analysis of word frequencies in old Kannada or Halegannada, using the corpus provided by the three texts *Kavirajamarga, Voddaradhane,* and *Pampasatakam*, and pointed out that the statistics are inadequate. As it happens, a number of subsequent works have criticised scaling laws [20] such as this on very similar grounds. The importance of examples and counterexamples stems primarily from the fact that the arguments are applied in reverse, namely, that if natural languages have characteristic patterns that are statistical, then any "text" with the same statistical patterns can be construed as a language and analysed as such. Indeed, some recent efforts to understand the organisation of DNA sequences proceed from this premise [18].

Either because of the interest in numismatics, or from his statistics work in general, DDK was led to consider the problem of computationally processing large amounts of data. The paper "Statistics in Function Space" [21] resulted from this research. The data was likely to be "high dimensional", namely, it had dependence on a large number of variables, and the question was whether there was some way of throwing out the inessential, and finding a "low dimensional" representation that would capture the main features of interest. This approach is used extensively in a number of areas of application: what was needed was a mathematical way of separating the wheat from the chaff.

Kosambi's work was published in the *Journal of the Indian Mathematical Society* in 1943, and regrettably, not followed up in any more internationally known forum. Today the method he proposed goes under the name of the Karhunen-Loève expansion, although Karhunen's work appeared in 1947 and Loéve's in 1948. There has even been a serious suggestion that this be renamed the Kosambi-Karhunen-Loève expansion [22]. Be that as it may, this is a widely used and very useful technique, whereby the main features of a problem, its so-called principal components can be extracted via the technique of proper orthogonal decomposition. As a tool for analysis, this is powerful, although there is considerable freedom in choosing the functions that constitute the essential modes.

That this work was motivated by Kosambi's studies in statistics, and the fact that he wrote this while he was also examining the statistics of ancient coins were perhaps not merely coincidental. In a lecture given on the occasion of Kosambi's birth centenary, Bhattacharjee [23] contends that

> the kind of thought process that guided Kosambi in his work on the proper orthogonal decomposition also dominated his study of history. [...] The resemblance with the proper orthogonal decomposition is striking. The essential factors need to be picked out in history, the correct set has to be chosen in the decomposition. Reading the book is not enough for history, experience is essential in deciding how many variables to keep in the decomposition.

On Map Distance

The other contribution of Kosambi's that has lived on is his work in genetics, on what is termed the map distance. Already in the early part of the 20th century, a quantitative approach to genetics was under way with the work of Thomas Hunt Morgan and others who had begun systematic studies of mutations in the common fruit fly. Morgan's discovery that variability from generation to generation originated in the redistribution of genes on chromosomes was seminal, and has led to the idea of genetic linkage, that groups of genes may move together when germ cells, namely, eggs or sperm, are formed. Morgan's hypothesis was that by examining the differences in crossover frequencies between linked genes, one might be able to estimate the distance between the genes on the chromosomes. This led to the definition of the so-called map distance, and the first estimate of this quantity was by J B S Haldane, in 1919.

Although Kosambi probably had no prior knowledge of genetics, his interest was sparked by the fact that Haldane's work was empirical and only approximate, while the mathematical statement of the problem allowed for a cleaner solution. There was also a thread of statistics that ran through this work, since many of the experimental observations on which the map distance was based involved estimates. The mapping function that Kosambi proposed was very general, and allowed for the fact that recombination events at different points on the chromosome were not necessarily independent, especially when the positions were nearby. In the light of our current knowledge of DNA structure this is a very natural allowance that must be made, but at the time when these proposals were made, the structure of genes – or indeed that of DNA – was largely unknown. "The Estimation of Map Distance from Recombination Values" [24] was published in the *Annals of Eugenics* (now the *Annals of Human Genetics*) in 1944, and it continues to be widely cited and used even to this day.

By 1946, he had fallen out with the authorities at Fergusson College, and had moved to the newly established TIFR Bombay. In January 1947 he was president of the Mathematics Section of the Indian Science Congress that was held in Delhi, and his presidential address "Possible Applications of the Functional Calculus" [25] is a discursive review of his ideas on function spaces and the proper orthogonal decomposition [26].

In 1948-49, DDK spent a long period of time in the US, first as visiting professor at the University of Chicago where he lectured on tensor analysis, followed by visits to the Institute for Advanced Studies in Princeton (where he met Einstein) and the Massachusetts Institute of Technology in Cambridge. His return to India was via London, where his interests in history brought him into contact with A L Basham. The 1950s saw DDK flourish as a historian, with the publication of his *Introduction to the Study of Indian History* [27] and numerous other works. He continued to apply statistics to a number of problems, but increasingly, his interest turned to prime numbers. This obsession was a dominant theme in his later years. Indeed, when he died he had just completed a manuscript, "Prime Numbers" (this has unfortunately been irretrievably lost [28]). Owing to differences in style as well as differences in priorities, he had a major rupture with the leadership at the TIFR where his appointment was not renewed after 1962. Although he continued to work, both in mathematics and in history, for the next two years he was without a formal position.

Pure Mathematics

This changed in 1964 when he was appointed a CSIR Emeritus Professor attached to the Maharashtra Vidnyanvardhini in Pune. The uncertainties and the acrimony that surrounded these years made him bitter, and when viewed from the outside after the passage of time, a trifle quixotic. He published a number of papers [29] under the pseudonym S Ducray [30], and chose to publish results that were potentially of tremendous significance in pure mathematics – a proof of the Riemann hypothesis [31] no less – in the *Indian Journal of Agricultural Statistics* [32]. At best, this choice of journal was viewed with perplexity. However, the results could not be substantiated and his mathematical reputation suffered as a consequence. In a posthumous review of this paper that appeared in the *Mathematical Reviews of the American Mathematical Society*, the Hungarian mathematician A Rényi generously interpreted DDK's unsuccessful attempt to prove the Riemann hypothesis as the statement of a new and independent conjecture, but was candid enough to assert that neither hypothesis could be proved or disproved using probabalistic methods [33].

In many ways Kosambi was temperamentally more a statistician than a pure mathematician; his appreciation of a proper use of statistics runs through many of his papers, and underlies his main contributions in science as well as in historiography and numismatics. He paid particular attention to the correct interpretation of statistical results, and emphasised this in several of his papers. Specific contributions such as the Kosambi distance and the principal orthogonal decomposition have stood the test of time, but in the end, Kosambi's main contribution was to emphasise a certain rigour in data analysis, whether it pertained to science, to genetics, to coins or to literature, and to make prescient and lateral inferences from such rigorous analysis.

In this sense, he was a member of an elite group of intellectuals who believed that a mathematical approach was valuable even when discussing and analysing phenomena that were typically classified as too "human", as not subject to the tools of analytic theories. Today these are in a sense commonplace, comprising as they do the core of complex systems studies where human behaviour itself is analysed via game theory, societies are modelled as groups of interacting agents, and language is analysed as an algebra. All these approaches, associated now with the names of von Neumann, Nash, Chomsky, Tinbergen, Simon, Arrow, and others, had their roots in the 1940s and the 1950s. In that sense, Kosambi was born at the right time, and it is fortunate that the unique combination of his diverse interests and his formidable intellect came together very naturally to find full expression in revitalising the study of Indian history.

NOTES AND REFERENCES

[1] For brevity and convenience, in this article I will refer to Damodar Dharmananda Kosambi as DDK or just Kosambi.

[2] A Gupta and K Haydock, *Bright Sparks* (New Delhi: Indian National Science Academy, 2009), pp 98-102.

[3] "Precessions of an Elliptical Orbit", *Indian Journal of Physics*, 5, 359-64 (1930).

[4] "Numismatics as a Science", *Scientific American*, March 1965.

[5] DDK used mathematics and statistics not only to bolster historical evidence, particularly in the field of numismatics, but also to debunk claims of extrasensory perception (ESP), in actuarial studies, in philology and in analysing trade and commerce.

[6] Kosambi studied differential geometry with William C Graustein (1888-1941) who was on the Mathematics faculty at Harvard. In later years, DDK was to serve on the committee that selected the Fields medalists, and in writing about that in the *Proceedings of the American Philosophical Society*, 137, 268-72 (1993), Garrett Birkhoff makes reference to "Graustein's student D D Kosambi".

[7] In a moving obituary published in *The Mathematics Student*, 12, 116-20 (1945) Kosambi discusses Birkhoff's mathematical contributions and the role he played in the development of American mathematics. Although he is frank in his appraisal, "Birkhoff's own lectures left very much to be desired even twenty-five years later. Those of his lectures that the advanced students did understand were invariably considered by them the most inspiring that they had ever heard, but usually he lost himself, the subject, and the audience in his own latest brainwave which might have developed that very morning between the breakfast table and the lecture room", he continues: "But the moment he began to talk about mathematics, the others (no matter what their specialty) quietly stopped their own chatter to listen. It was impossible not to love such a teacher."

[8] A Weil, *The Apprenticeship of a Mathematician* (Basel: Birkhäuser, 1992).

[9] "On a Generalisation of the Second Theorem of Bourbaki", *Bulletin of the Academy of Sciences*, UP 1, 145-47 (1931). Kosambi thanks Weil for making him aware of the "important work" of this Bourbaki. The French group eventually chose the initial N (Nicolas) for Bourbaki rather than the D (Damodar?) given by Kosambi.

[10] M Mashaal, *Bourbaki: A Secret Society of Mathematicians* (American Mathematical Society, Providence, 2006).

[11] Starting with "On the Existence of a Metric and the Inverse Variational Problem", *Bulletin of the Academy of Sciences*, UP 2, 17-28 (1932), DDK developed the idea in a number of papers, including "Parallelism and Path-spaces", *Mathematische Zeitschrift*, 37, 608-22 (1933) and "Path-spaces of Higher Order", *Quarterly Journal of Mathematics* (Oxford), 7, 97-104 (1936) and so on. In the 1950s he was on the editorial board of the Japanese journal, *Tensor* (New Series) wherein he published "The Metric in Path-space", *Tensor* (New Series), 3, 67074 (1954), possibly his final paper on the topic.

[12] The papers "Geometrie differentielle et calcul des variations", *Rendiconti della Reale Accademia Nazionale dei Lincei*, 16, 410-15 (1932) and "Affin-geometrische Grundlagen der Einheitlichen Feldtheorie", *Sitzungsberichten der Preussische Akademie der Wissenschaften, Physikalisch-mathematische klasse*, 28, 342-45 (1932) were communicated to the journals by T Levi-Civita, while, "Parallelism and Path-spaces", *Mathematische Zeitschrift*, 37, 608-22 (1933) was communicated by E Cartan, who wrote a commentary, "Observations sur le memoire precedent", *Mathematische Zeitschrift*, 37, 619-22 (1933) that followed DDK's paper.

[13] The paper, "The Method of Least-squares", *Journal of the Indian Society of Agricultural Statistics*, 11, 49-57 (1959) originally appeared in Chinese in the Academica Sinica journal, *Advancement in Mathematics*, 3, 485-91 (1957).

[14] "The Emergence of National Characteristics among Three Indo-European People", *Annals of the Bhandarkar Oriental Research Institute*, 20, 195-206 (1940).

[15] "A Note on Two Hoards of Punch-Marked Coins Found at Taxila", *New Indian Antiquary*, 3, 156-57 (1940).

[16] Indeed, the work in the *New Indian Antiquary was also Reviewed in Current Science*, 7, 345-46 (1941) by KAN (possibly K A Nilakanta Sastri, the pre-ëminent historian of South India) who declares while admitting to not understanding the mathematics of it all, that this work "is important, and one hopes that the Director General of Archaeology, the museums of the country and individual owners of coin-cabinets will give him [Kosambi] all the aid he needs for carrying his researches further."

[17] The papers that DDK published here include "A Statistical Study of the Weights of the Old Indian Punch-Marked Coins", *Current Science*, 9, 312-14 (1940); "On the Weights of Old Indian Punch-Marked Coins, *Current Science*, 9, 410-11 (1940); "A Bivariate Extension of Fisher's Z-test", *Current Science*, 10, 191-92 (1941*);* "Correlation and Time-Series", *Current Science,* 10, 372-74 (1941); "On the Origin and Development of Silver Coinage in India", *Current Science*, 10, 395-400 (1941); "The Effect of Circulation Upon the Weight of Metallic Currency", *Current Science*, 11, 227-31 (1942); "A Test of Significance for Multiple Observations", *Current Science*, 11, 271-74 (1942).

[18] A comprehensive collection of articles on Zipf's law has been maintained by Wentian Li at the website http://www.nslij-genetics.org/wli/zipf/. The applicability of the law to Indian languages has been re-investigated recently, by B D Jayaram, and M N Vidya, "Zipf's Law for Indian Languages", *Journal of Quantitative Linguistics*, 15, 293-317 (2008).

[19] "On Valid Tests of Linguistic Hypotheses", *New Indian Antiquary*, 5, 21-24 (1942). DDK uses Kanarese rather than Kannada and Halagannada rather than Halegannada.

[20] See, e g, W Li, "Zipf's Law Everywhere", *Glottometrics*, 5, 14-21 (2003).

[21] "Statistics in Function Space", *Journal of the Indian Mathematical Society*, 7, 76-88 (1943).

[22] G Kallianpur, *Bulletin of the American Mathematical Society* 34, 43-48 (1997).

[23] J K Bhattacharjee, "Turbulence" (Lecture given on the occasion of the Kosambi centenary, University of Calcutta, 2007).

[24] "The Estimation of Map Distance from Recombination Values", *Annals of Eugenics*, 12, 172-75 (1944). While commenting on the main result, DDK's eclectic scholarship, as well as his sharpness, comes through as he says of the formula: The similarity of this with the velocity addition formula in the special theory of relativity *should not be made the basis of more bad philosophy* (italics added).

[25] "Possible Applications of the Functional Calculus", *Proceedings of the 34th Indian Science Congress. Part II: Presidential Addresses*, 1-13 (1947).

[26] In [25], Section 8, Kosambi gives the following examples of where the functional calculus techniques would apply. If average temperature curves are available for any range or period, is it possible to say whether two samples from two different places differ materially? Or do two skulls found by the archaeologist or anthropometrician in two different places differ significantly? The need for a mathematical technique to decide questions of this form are suggestive of how his interests in one area inspired work in the other.

[27] *Introduction to the Study of Indian History* (Popular Book Depot: Bombay, 1956).

[28] Apparently DDK mailed the only copy he had to the publishers Routledge and Kegan Paul, but all efforts to trace that or a second copy that may have been given to H Bhabha have been in vain.

[29] The four papers published by D D Kosambi as S Ducray were (a) "A Note on Prime Numbers", *Journal of the University of Bombay*, 31, 1-4 (1962), (b) "Normal Sequences*I", Journal of the University of Bombay*, 32, 49-53 (1963), (c) "Probability and Prime Numbers", *Proceedings of the Indian Academy of Sciences*, 60, 159-64 (1964), and (d) "The Sequence of Primes", *Proceedings of the Indian Academy of Sciences*, 62, 145-49 (1965). Of these, (b) and (c) were reviewed in *Mathematical Reviews*.

[30] The Kosambi family had a dog name Bonzo who was fondly known as *Dukkar* (meaning pig in Marathi) on account of his being somewhat rotund. Given DDK's sense of the mischievous, S Ducray was the French-sounding version of the dog's name, the initial being for *Svana*, namely, the Sanskrit for dog. That their combination echoed schweinhund could hardly have escaped DDK who knew German well enough to teach the language at the Banaras Hindu University. In the papers written as S Ducray, DDK is thanked profusely.

[31] The Riemann hypothesis is considered by most mathematicians to be the most important unresolved problem in pure mathematics. This states that the Riemann zeta-function $\zeta(s) = \Sigma 1/n^s$, where *s* is a complex number and the sum is over all the integers has "nontrivial" zeros only when the real part of s has the value 1/2. This is one of the so-called Millennium Prize problems for which a grand cash award has been announced by the Clay Mathematical Institute.

[32] "Statistical Methods in Number Theory", *Journal of the Indian Society of Agricultural Statistics*, 16, 126-35 (1964).

[33] *Mathematical Reviews* MR0217024 (36 # 119).