THE GENERALIZED TIME-DELAYED HÉNON MAP: BIFURCATIONS AND DYNAMICS

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Received January 27, 2012; Revised May 23, 2012

We analyze the bifurcations of a family of time-delayed Hénon maps of increasing dimension and determine the regions where the motion is attracted to different dynamical states. As a function of parameters that govern nonlinearity and dissipation, boundaries that confine asymptotic periodic motion are determined analytically, and we examine their dependence on the dimension $d$. For large $d$ these boundaries converge. In low dimensions both the period-doubling and quasiperiodic routes to chaos coexist in the parameter space, but for high dimensions the latter predominates and prior to the onset of chaos, the systems exhibit multistability. When the nonlinearity parameter is varied, the dimension of chaotic attractors in the systems changes smoothly with increasing number of non-negative Lyapunov exponents.

Keywords: High dimension; diffeomorphism; dissipative map; normal form coefficient; limiting curves; hyperchaos.

1. Introduction

Over the past few decades, the behavior of low-dimensional nonlinear iterative maps and flows has been extensively studied and characterized, in particular with reference to the creation of chaotic dynamics [May, 1976; Li & Yorke, 1975; Hénon, 1976]. The various scenarios or routes to chaos in such systems are by now fairly well known [May, 1976; Arnold, 1965; Newhouse et al., 1978; Rand et al., 1982]. The motion in higher-dimensional systems — for instance, the dynamics of attractors with more than one positive Lyapunov exponent and the bifurcations through which they have been created — has not been studied in as much detail even in relatively “simple” systems [Albers & Sprott, 2006; Sprott, 2006; Baier & Klein, 1990].

Our goal in the present work is to explore the transition from low- to high-dimensional dynamics in a generalized Hénon map [Hénon, 1976]. This time-delayed iterative map with a single quadratic nonlinear term also contains the time-delayed Hénon maps introduced by Sprott [2006] and Baier [Baier & Klein, 1990] as special cases. The phenomenon of multistability in high-dimensional maps has been addressed in recent studies [Richter, 2008; Sprott, 2006] and in particular, multistability near the onset of chaos was studied by Sprott [2006]. While the principal focus in the work of Baier [Baier & Klein, 1990] was the study of hyperchaos, we find that the present generalization of the quadratic map also results in a system that is hyperchaotic, while exhibiting multistability.
Furthermore, we characterize the different bifurcations that occur as parameters are varied.

The organization of this paper is as follows. In the next section, the generalized Hénon map is described and a detailed analysis of the local bifurcations of the elementary fixed points is presented. In Sec. 3, the emergence of chaotic and hyperchaotic attractors is described, and multistability and global bifurcations are discussed. The system exhibits an interesting transition as \( d \to \infty \), when the regions of bound motion in parameter space change to an unbounded region. The paper concludes with a discussion and summary in Sec. 4.

2. The Generalized Hénon Map

The time-delayed Hénon map considered here is given by

\[
x_n = 1 - ax_{n-k} - (1 - \nu)x_{n-d}.
\]

The four parameters are, respectively, the nonlinearity \( a \) which is also usually the bifurcation control parameter, the dissipation is determined by \( \nu \in [0, 2] \), and the larger of the positive integer parameters \( d \) and \( k \) decides the dimensionality of the map. For the quadratic nonlinearity considered, we take \( 1 \leq k < d \), so that the map is a diffeomorphism. For \( \nu = 1 \) the map ceases to be a diffeomorphism for any \( k \). Effectively, therefore, this map has linear feedback from previous time steps and nonlinear feedback from the \( k \)th earlier step. When \( k = 1 \), this reduces to the system studied earlier by Sprott [2006], whereas when \( k = d - 1 \) this map reduces to the one introduced by Baier [Baier & Klein, 1990]. For this latter case as has been discussed previously [Baier & Klein, 1990], the nonlinear delay term implements stretching and folding in \( d-1 \) dimensions, and thus as the phase space dimension increases there is a concurrent increase in the number of positive Lyapunov exponents when the dynamics is hyperchaotic [Baier & Klein, 1990].

Since the Jacobian matrix \( J \) of the quadratic Hénon maps considered here has a constant nonzero determinant Eq. (2) (for \( \nu \neq 1 \)), these are so-called Keller or Cremona maps [Esen, 2000]. Such maps are elements of the group of polynomial automorphisms, and while in two dimensions these are known to be conjugate to a composition of affine and elementary maps [Dullin & Meiss, 2000], similar results in higher dimensions are not available [Esen, 2000; Gonchenko et al., 2006, 2010]. Explicitly, the Jacobian matrix elements are given by

\[
J_{ij} = \begin{cases} 
1 & j = i - 1, \quad d \geq i \geq 2 \\
-2ax^{(k)} & i = 1, \quad j = k \\
-(1 - \nu) & i = 1, \quad j = d \\
0 & \text{otherwise},
\end{cases}
\]

where \( x^{(k)} \) is the \( k \)th coordinate of the map (1) written in expanded form, and the determinant \( |J| = 1 - \nu \).

In the present work, we consider the case \( k = d - 1 \), and undertake a numerical study of the dynamics of Eq. (1). When \( 2k = d \) there is an important simplification of the system, essentially factorizing it into \( k \) copies of the quadratic system [Bilal, 2013]. While Baier and Klein [1990] introduced this case as an example exhibiting hyperchaos in \( d \) dimensions, here we investigate the dynamics as function of \( a, \nu \) and \( d \).

For any dimension, the map has a fixed point, namely period-1 dynamics. The region where the fixed point is stable can be determined quite simply by solving the corresponding algebraic equations. These boundary curves can be obtained analytically for \( d = 3 \) while for \( d = 4 \) and 5 a center-manifold analysis can be carried out. Finally, in the limit \( d \to \infty \) the boundary of the region in parameter space where the fixed point is stable converges. We also investigate the route to hyperchaos in this family and generalize the idea of a smooth transition from chaos to hyperchaos; in addition, we numerically establish that the map can be decomposed into submaps [Harrison & Lai, 2000].

The stability of a given periodic point can be determined from the nature of the eigenvalues of Jacobian matrix, namely the roots of the associated characteristic polynomial. From Eq. (1), it can be seen that the solutions for the fixed point are

\[
x^i_\pm = \frac{-(2 - \nu) \pm \sqrt{(2 - \nu)^2 + 4a}}{2a},
\]

and these roots are born at a fold bifurcation [Kuznetsov, 1998] for \( a = -\frac{1}{2}(2 - \nu)^2 \). Of the two roots \( \{x^-_i\} \) is always unstable while \( \{x^+_i\} \) is stable for \( 1 \leq \nu \leq 2 \) (although the range of stability will also depend on the value of the parameter \( a \)). This latter root is unstable when \( 0 \leq \nu \leq 1 \) and the bifurcation leading to the stability of \( \{x^+_i\} \) depends...
on the dimension $d$. There is thus a region of stable fixed point behavior for an appropriate range of $\nu$ and $a$. The bifurcation loci can be computed by evaluating the roots of the characteristic polynomial, namely

$$p(\lambda) = \lambda^d + 2ax + \lambda^{d-k} + (1-\nu),$$

(4)

with $k = d - 1$. At a flip bifurcation, $\lambda = -1$, at the fold bifurcation $\lambda = 1$, and at the Neimark–Sacker (NS) bifurcation $\lambda = e^{\pm i \phi}$ [Kuznetsov, 1998]. Equation (4) then defines a curve in the $a-\nu$ plane along which these bifurcations occur. The flip and NS boundaries depend on $d$ and these can also be calculated using a combination of analytical and numerical methods,

$$L_{\text{fold}}(d) : a = -\frac{1}{4}(2 - \nu)^2$$

$$L_{\text{flip}}(d) : a = \begin{cases} 
\frac{3}{4}(2 - \nu)^2, & d = 2n \\
\frac{1}{4}(3(1-\nu)^2 - 3 + 2\nu), & d = 2n + 1, \ n = 1, 2, \ldots
\end{cases}$$

We find that for general $d$ there are four branches of the bifurcation boundary. On two of these boundary branches, the NS bifurcation occurs, the flip and fold bifurcations occurring respectively on the others. For large enough $d$ the period-1 region is found to be bounded by a set of limiting curves that do not depend on whether $d$ is odd or even and are shown in Fig. 1(c). These are given by

$$L_{\text{T}} : a = \frac{1}{4}(2 - \nu)^2$$

$$L_{\text{R}} : a = \frac{3}{4}(2 - \nu)^2$$

$$L_{\text{L}} : a = \frac{1}{4}(3(1-\nu)^2 - 3 + 2\nu)$$

$$L_{\text{RL}} : a = \nu \left( 1 - \frac{1}{d} \right),$$

(5)

the superscripts $T, B, L$ and $R$ denoting top, bottom, left and right segments of the boundaries. Two of these limiting curves are found to overlap with the

![Fig. 1](image)

(a) The region of period-1 dynamics for $d = 100$, (b) region of bound dynamics for $d = 100$, (c) the asymptotic curves corresponding to period-1 dynamics as obtained semi-empirically, and (d) the area of bound dynamics $\Delta_{\text{bound}}$ as a function of dimension $d$. 

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Fig. 2. Region where the normal form coefficient is negative (black) and positive (red) for the Neimark–Sacker bifurcation. The dark green curves represent the NS boundaries in (a) \( d = 4 \), and (b) \( d = 5 \). The blue region in (a) indicates that \( c(0) \neq 0 \) on the flip bifurcation (green) curve: see text for explanation.

boundaries in Eq. (5) while the remaining two correspond to the NS bifurcation (as observed in the spectrum of roots of the characteristic polynomial). Since the Jacobian matrix has constant determinant, there is a constraint on the distribution of eigenvalues [and hence the roots of Eq. (4)] and this results in a saturation of the bifurcation boundaries as the dimension \( d \) is increased.

It is possible to obtain the normal form coefficients on the center manifold [Kuznetsov, 1998] for the NS and flip bifurcations. Shifting Eq. (1) to coordinates such that the origin (in \( a - \{x^1\} \) space for \( \nu \) fixed) is nonhyperbolic (i.e. the bifurcation point), a Taylor expansion about this fixed point gives the mapping

\[
x \rightarrow Jx + F(x), \quad x \in \mathbb{R}^d
\]

where \( F(x) \) contains the \( O(x^2) \) terms. Through detailed algebraic analysis, following the methods outlined in [Kuznetsov, 1998], it is easy to show that for the flip bifurcation, the restriction of a map to its center manifold is equivalent to the one-dimensional normal form,

\[
\dot{\xi} = -\xi + c(0)\xi^3.
\]

Similarly for the NS case, the restriction on the center manifold is equivalent to the normal form

\[
\dot{z} = e^{i\theta_0}z(1 + d(0)|z|^2),
\]

where the real number \( \omega(0) = \text{Re}d(0) \), determines the direction of bifurcation. Although, in general, it is cumbersome to obtain the exact expressions for the normal formal coefficients, in the present case since derivatives of order three and higher vanish, the coefficients can be easily computed numerically using the projection method [Kuznetsov, 1998]. At the appropriate bifurcation boundaries the normal form coefficients must have \( c(0) \neq 0 \) and \( \omega(0) < 0 \) for the flip and the NS bifurcations, respectively. That these conditions are satisfied in the present instance is shown in Fig. 2 for the cases of \( d = 4 \) and \( d = 5 \). Period-1 dynamics thus loses stability by different bifurcations along the different segments of the boundary.

3. High-Dimensional Dynamics

In this section we study the bifurcations of the period-1 attractor for higher embedding dimensions. The two-dimensional case has been studied extensively [Sonis, 1996; Feudel & Grebogi, 2003].

3.1. Bound dynamics

For larger \( d \) there is a limited region in parameter space that supports bound motion which we explore numerically. Keeping the dissipation parameter \( \nu \) fixed, \( a \) is varied starting from the period-1 region, with fixed initial conditions \( \{x^1_0, x^2_0, x^3_0, \ldots\} \), and taking the initial conditions for next \( a \) from the attractor at previous \( a \). This helps to delineate the region of bound motion in the \( a-\nu \) plane, the case \( d = 100 \) is shown in Fig. 1(b). We calculate the area of bound motion (\( \Delta_{\text{bound}} \)) numerically as a function of dimension \( d \), and as can be seen in Fig. 1(d), it converges with increasing dimension, \( d \to \infty \).

3.2. Route to hyperchaos

There are several routes to chaos that can be observed in this particular family of maps. There is a transition to chaos via quasiperiodicity, as shown...
Fig. 3. The largest Lyapunov exponent for two different values of dissipation $\nu = 1.1$ (black) and $\nu = 0.9$ (red) for (a) $d = 5$, (b) $d = 10$, (c) $d = 50$, and (d) $d = 100$. The standard Gram-Schmidt procedure was used to calculate the Lyapunov exponent spectrum. The trajectory length was taken to be $10^4$.

in Fig. 3 for all $d$. The flip bifurcation leads to chaos via period doublings in the logistic map and the two-dimensional Hénon map. Along this route which can be seen upon varying $a$ for large $d$, the period-doubling cascade appears to be truncated after a finite number of doublings.

Since most $d - 1$ of the Lyapunov exponents can be positive [Baier & Klein, 1990] the question of how the hierarchy of chaos sets in is of interest. The manner in which the attractors evolve can be examined, for instance, by computing a fractal dimension of the underlying attractor as a function of relevant system parameters. We compute the Lyapunov dimension as a function of the nonlinearity parameter $a$ and the results are shown in Fig. 4. The transition from chaos to hyperchaos is smooth; the dimension of the attractor evolves gradually as the nonlinearity parameter is gradually increased.

We further examine the chaos to hyperchaos transition by considering the analysis introduced by Harrison and Lai [2000] in their study of two mutually coupled logistic maps. In a study of two coupled logistic maps, they noted that the largest subsystem Lyapunov exponent passes through zero smoothly at this transition. Here we consider the generalized Hénon maps coupled to the quadratic map Eq. (10),

$$
\begin{align*}
    x_{n+1} &= 1 - ax_{n-d+2} - (1 - \nu)x_{n-d+1} + \epsilon y_n \\
    y_{n+1} &= \alpha - y_n^2 + \epsilon x_n
\end{align*}
$$

Fig. 4. The Kaplan-Yorke dimension for (a) $d = 4$. The corresponding few largest Lyapunov exponents are plotted in (b), and as can be seen, there are no abrupt transitions as the number of positive Lyapunov exponents increases. The transition to higher-dimensional chaos is thus smooth in this system.
Fig. 5. A smooth transition to hyperchaos for $d = 3$, $k = 2$ in Eq. (1) coupled to the logistic map. Note that the effective dimension of this system is four. The dissipation is fixed $\nu = 1.1$.

with $\alpha = 1.3$, $\epsilon = 0.05$. Taking $d = 3$ we observe that upon variation of the nonlinearity parameter of the driven map, namely $\alpha$, the third-largest Lyapunov exponent crosses zero smoothly, marking a transition of the whole system from two positive Lyapunov exponents to three positive Lyapunov exponents. This is shown in Fig. 5 and as can be seen the transition in the Hénon system with $d = 4$ is very similar; see Fig. 4. This latter case can be understood as follows: once the dynamics in the Hénon system becomes hyperchaotic, it drives the logistic map hyperchaotically. When the parameter of this map is varied, an additional Lyapunov exponent of the composite system becomes positive. Thus, in general the system can be viewed as being composed of $(d - 1)$ independent subsystems driving each other in the sense described above at the point of transition to high-dimensional chaos.

3.3. Multistability

Note that at the same time, there can be multistable dynamics which can be detected via the useful order-parameter introduced by Sprott [2006], namely the mean square deviation from a reference point, $\psi_r$,

$$\langle r^2 \rangle = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} (\psi_n - \psi_r)^2, \quad (11)$$

where $\psi_n$ represents the time series under consideration originating from a deterministic dynamical system (as in our case) and $\psi_r$ is chosen appropriately. At a given set of system parameters, this order parameter will be distinct for each attractor and thus if different initial conditions yield $\langle r^2 \rangle$ values that cluster around distinct and different values, then multiple attractors must coexist.

We set $\psi_n = x_1^n$ and for the choice of $\psi_r = 0$ a plot of $\langle r^2 \rangle$ as a function of the parameter $a$ is given in Fig. 6. For given $d$, this order parameter is evaluated for an ensemble of initial conditions at different values of the nonlinearity, and a distinct regime of multistable dynamics becomes apparent.

Fig. 6. The root mean square displacement as a function of $a$ for $\nu = 1.1$ and (a) $d = 4$, (b) $d = 5$, (c) $d = 6$, and (d) $d = 11$. The multivaluedness of this measure indicates multistability. The arrows mark the point of transition to chaos as determined by the largest Lyapunov exponent.
in the region immediately preceding the transition to chaotic motion when the order parameter becomes multivalued. Taken in conjunction with the results of Sprott [2006] this observation in the present hyperchaotic map indicates that the transition to chaos in high-dimensional systems is preceded by multistability, independent of whether the map under consideration is hyperchaotic or chaotic.

4. Summary

In this paper we have introduced a generalized time-delayed Hénon map with nonlinear feedback from \( k \) earlier steps and linear feedback from \( d \) earlier steps. Bifurcation analysis has been possible in a limited region of the parameter space. In particular, fixed point dynamics can be lost through the fold, flip and the Neimark-Sacker bifurcations. Analogous forms were determined for these boundaries, and the fold and NS bifurcation curves were found to depend on the dimension \( d \) being odd or even. With increasing dimension, the region of period-1 dynamics along with that of bound motion was found to converge in parameter plane.

The attractor of the dynamics evolves smoothly when the Lyapunov exponents become positive. Subsequent to the transition to hyperchaos, further increase in the number of positive Lyapunov exponents does not appear to alter the basic morphology of the attractor [Harrison & Lai, 2000]. For large dimensions, the dominant route to chaos is found to be via quasiperiodicity independent of \( d \) being odd or even. Near the transition to chaos multiple attractors coexist, and this appears to be a common feature of the transition to chaos in high-dimensional systems [Sprott, 2006; Albers & Sprott, 2006].

Acknowledgment

S. Bilal would like to thank the CSIR, India for support through a Junior Research Fellowship.

References


