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Delay-coupled discrete maps: Synchronization, bistability, and quasiperiodicity

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ABSTRACT

The synchronization transition is studied in delay-coupled logistic maps. For low coupling, in-phase and out-of-phase synchronous dynamics coexist, and with increasing coupling there is a regime of quasiperiodicity before eventual attraction to a fixed point at a critical value of coupling that depends on the nonlinearity. The presence of a region of asynchrony separating two synchronized regimes—termed *anomalous* behaviour—has been observed earlier in continuous systems and is shown here to occur in delay mappings as well. There are regions of in-phase, anti-phase, and out-of-phase dynamics of periodic as well as chaotic attractors.

1. Introduction

Amplitude death [1] and synchronization [2] are among the most extensively studied of the various dynamical phenomena that can arise when two nonlinear systems are coupled. In the former case, two oscillatory systems, when coupled drive each other to fixed points, resulting in a loss of oscillation (and therefore amplitude *death*). In the latter case, the two systems continue to oscillate with unique responses to one another [2].

When dealing with coupled systems, most studies have taken the interaction to be instantaneous. It has often been pointed out that in order to properly treat many physical or biological systems—in which the signals that mediate the interaction have finite transmission time—the interaction could be time-delayed. Recent studies have probed the manner in which systems synchronize when there is time-delay in the coupling [3–5], and have also examined the nature of amplitude death [4–6] and synchronization [3,7,8] when there is delay.

In this Letter, we examine the dynamics of discrete mappings [9] coupled with delays. Earlier studies of either synchronization or amplitude death which have largely focused on time-continuous dynamical systems, namely flows. An advantage in studying mappings is that some aspects of the analysis become simpler, particularly with reference to multistability [10]. Although delay increases the dimensionality of the problem, unlike the case of flows, the system remains finite-dimensional in discrete delay mappings. We find that the dynamics can be periodic, quasiperiodic or chaotic as the coupling strength is varied, and above a critical coupling (which depends on the nonlinearity parameter) the dynamics goes to a fixed point attractor: this, effectively, is the analogue of amplitude death in flows. In the transition from periodic motion to a fixed point (which can also be considered a synchronized state), there is an intervening asynchronous regime where the motion is quasiperiodic; this behaviour has been termed *anomalous* in the sense that the transition is not uniformly in the synchronized regime [11,7].

Below we describe the model system of delay coupled maps studied here. Our results on different synchronization transitions are presented in Section 3, and this is followed by a summary and conclusion in Section 4.

2. The model system

We consider bidirectionally coupled maps, with variables denoted by x and y respectively,

$$x_{n+1} = (1-\beta) f(x_n) + \beta g(y_n),$$
$$y_{n+1} = (1-\beta) f(y_n) + \beta g(x_{n+1}). \tag{1}$$

In the present work we take $f(x)$ to be the logistic function, $\alpha x(1-x)$. The coupling is asymmetric in the delay, and the coupling function $g(.)$ is taken to be linear. Such maps (in the absence of delay) have been studied extensively in the past [12], especially

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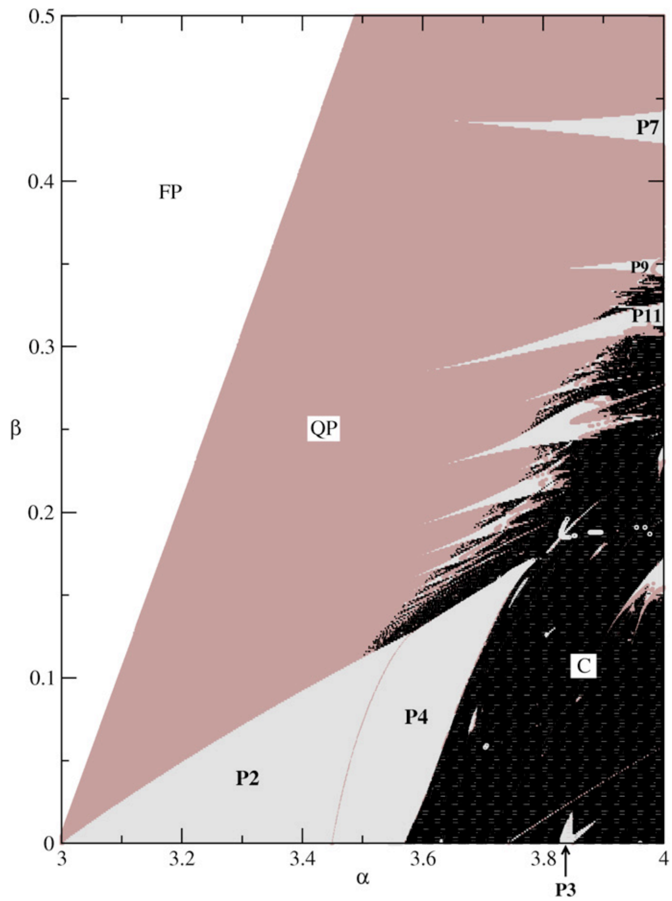

Fig. 1. (Color online.) Schematic phase diagram in (α, β) parameter space showing different regions of periodic—Pn (for $n = 2$, 3, 4, 7, 9, and 11) (light grey), chaotic—C (dark), quasiperiodic—QP (deep grey) dynamics. In the white region marked FP, the dynamics goes to a fixed point.

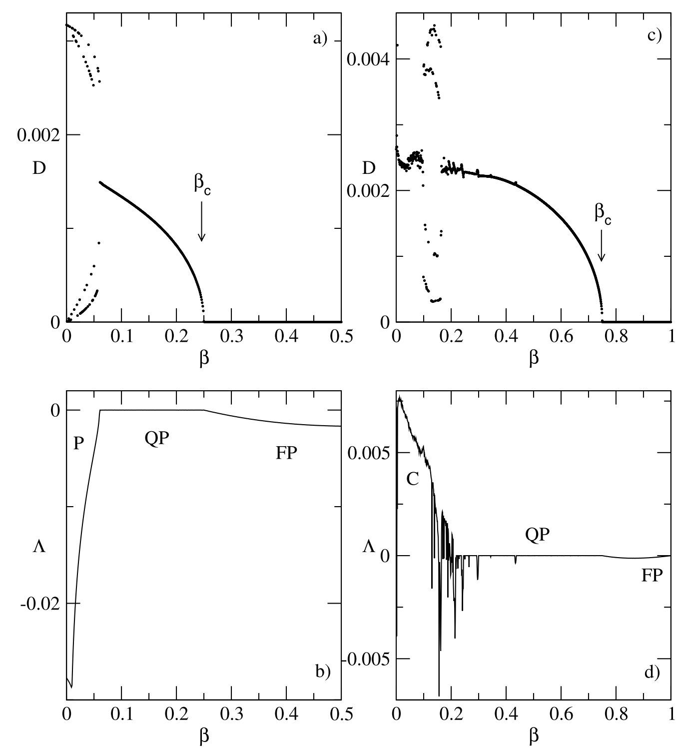

Fig. 2. (a) Average distance D between trajectories x and y for fixed parameter value $\alpha = 3.25$, (b) the Lyapunov exponents as a function of β. In (c) and (d), the corresponding results for $\alpha = 3.75$. β_c indicates the transitions from quasiperiodic motion to fixed point behaviour.

in the context of on–off intermittency [13], and multistability [14]. The control parameters are the nonlinearity α and the coupling strength β. (We find that the results are very similar with symmetric delay coupling, and we briefly discuss the case of nonlinear coupling in Section 3.)

In the numerical simulations, we start from random initial conditions in x and y. When complete synchronization occurs orbits coincide, and thus one order parameter for synchronization is simply the average distance between the trajectories,

$$D = \langle d \rangle = \frac{1}{N}\left\langle \sum_{i=1}^{N} |x_i - y_i| \right\rangle, \tag{2}$$

the average $\langle\cdot\rangle$ being taken over an ensemble of initial conditions. D clearly should vanish in the synchronized phase while $D > 0$ indicates a lack of complete synchrony. Other order parameters such as the largest Lyapunov exponent Λ, the average lagged-difference m (namely if $x_k = y_{k+m}$) and magnetization μ, the average value of the "local direction phase" [15,16] $S = \pm 1$ ($+1$ if $x_n > y_n$, else -1) over a trajectory of length N, may also be used to study the relative dynamics between two or more interacting systems. Here we only use the distance D and the Lyapunov exponents, and take N to be 10^4.

3. Synchronization transitions

When $\beta \neq 0$, the coupling can drive the system to a stable fixed point or to stable periodic/quasiperiodic dynamics, and depending on the value of α, the dynamics of the (uncoupled) logistic maps can be periodic or chaotic [17]. Thus these transitions to the stabilized synchronous dynamics can be approached either from the periodic or from the chaotic regime.

A schematic phase diagram in the $(\alpha–\beta)$ plane is shown in Fig. 1 which gives an idea of the different kinds of motion in the coupled system. Periodic dynamics occurs in the regions marked P (light grey), while quasiperiodic and chaotic motion occur in the regions marked QP (deep grey) and C (black) respectively. The white region corresponds to fixed point (FP) dynamics, and as can be seen, this region can be approached from either periodic or chaotic states via intermediate quasiperiodic motion. An earlier study of tent-maps coupled in a similar fashion [18] had noted the direct transition from chaotic dynamics to fixed points.

Shown in Fig. 2 are the order parameter D in (a)–(c) and the largest few Lyapunov exponents Λ in (b)–(d) respectively, the left and right panels being for different values of nonlinearity. When $\alpha = 3.25$ the uncoupled motion is stable period 2 and at $\alpha = 3.75$, the motion is chaotic. The curves in Fig. 2(a) and (c) are obtained from an average of 100 different initial conditions.

For $\alpha = 3.25$, as β is increased from zero (see Fig. 2(a)–(b)) there is first purely periodic motion (region P in Fig. 1). In this region there is bistability: depending on initial conditions, both in-phase and out-of-phase dynamics coexist, and typical trajectories are shown in Fig. 3(a) and (b) respectively. With further increase of β there is loss of multistability and the dynamics becomes quasiperiodic: the x and y motions have a mixed-phase relation that changes with time. This asynchronous regime is shown in Fig. 3(c), and the transient dynamics of the two variables x and y in the fixed point regime is shown in Fig. 3(d).

With further increase in the coupling strength there is a transition from the quasiperiodic unsynchronized state to a fixed point at a critical value of the coupling β_c. The average distance between the two trajectories goes to zero at β_c (which depends on α of course) continuously (Fig. 2(a)–(c)) as a power law

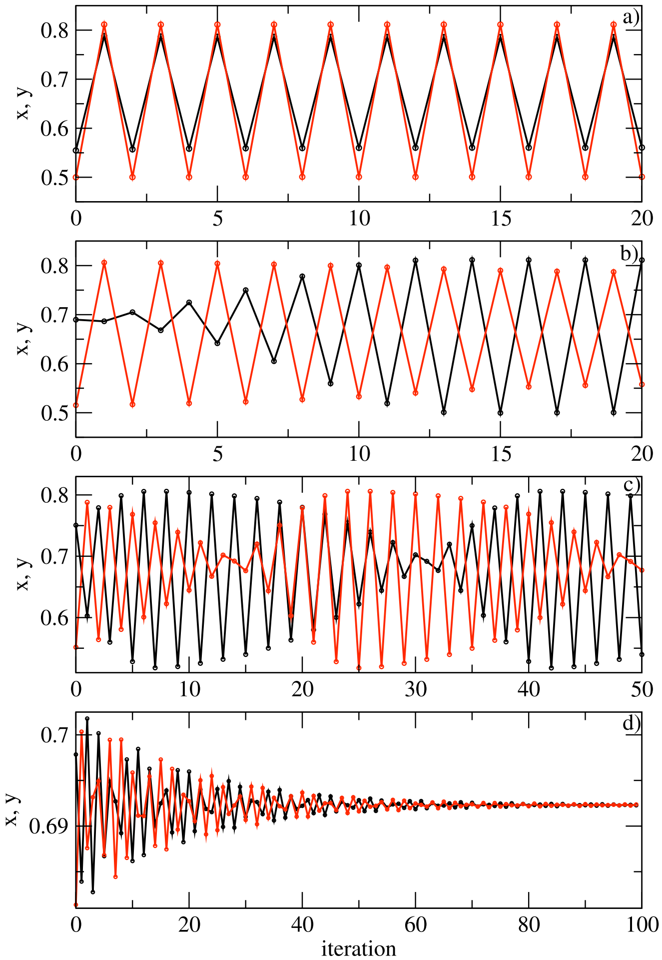

Fig. 3. (Color online.) Time series of x (solid line) and y (dashed line) for $\alpha = 3.25$ showing (a) in-phase motion and (b) out-of-phase dynamics at $\beta = 0.05$, (c) quasiperiodic behaviour for $\beta = 0.1$, and (d) transient dynamics in the approach to the fixed point at $\beta = 0.25$.

$$D \propto (\beta - \beta_c)^{\nu} \tag{3}$$

and the exponent ν appears to be ~ 0.5 independent of α and thus of β_c; see Fig. 4. Similar power-law behaviour was observed for an analogous transition in a family of unimodal maps with $f(x) = 1 - \alpha|x|^z$ for $z > 1$. When $z \leqslant 1$, the map has a cusp at the maximum and is non-differentiable, and this leads to a discontinuous transition [18]. A plot of the Lyapunov exponent as a function of β (see Fig. 2(b)–(d)) shows a change in the slope at the transition. We have studied this family of maps as well, and find that the scaling exponent ν appears to be independent of z for $z \geqslant 1$ as well.

We note that an asynchronous state intervenes between synchronized periodic motion and the synchronized fixed point. Such behaviour has been termed *anomalous* in the context of studies of phase disorder in coupled systems [11,7] since the subsystems go out of synchrony as the coupling is increased. The width of the anomalous regime of quasiperiodic motion increases as a function of both the parameters α and β (Fig. 1). From Figs. 1 and 2, it is evident that as the coupling strength is increased, the Lyapunov exponent starts decreasing. As in other instances of forcing and coupling, the period-doubling cascade is truncated and we have not been able to observe periods above 16 in our simulations; chaos is suppressed due to time-delay interaction. Some odd-order periodic orbits emerge due to delay in separate islands in the light grey regions that are interwoven within the quasiperiodic regime in Fig. 1.

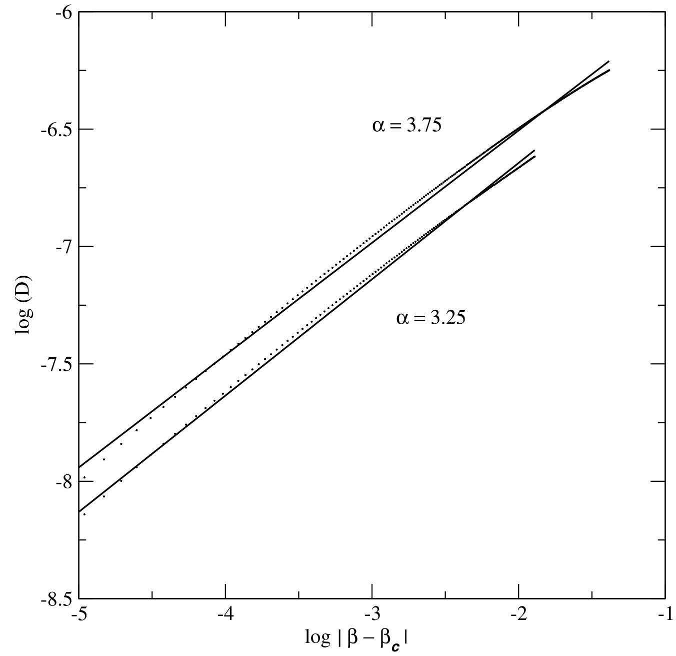

Fig. 4. Variation of D with $\beta - \beta_c$ for $\alpha = 3.75$ (star) and $\alpha = 3.25$ (circle). The dotted line has slope -0.5.

We have also considered the case of quadratic coupling, with the function $g(x)$ taken for convenience to be logistic as well as in other studies without delay [12]. With nonlinear coupling there are two transitions, first from asynchronous to completely synchronous dynamics which is then followed by the synchronization–desynchronization transition as above. If, further, the coupled systems are made nonidentical by changing the nonlinearity parameters, say, different regions of in-phase, anti-phase, and out-of-phase dynamics in both the periodic as well as chaotic regimes can be observed.

4. Summary and conclusions

In this Letter, we have studied a simple example of delay coupling between two logistic maps. The coexistence of different stable attractors at a given set of parameter values is a pervasive feature of delay-coupled maps, and in the present case there are large regions in parameter space where both in-phase or out-of-phase synchronous motions coexist. With increasing coupling strength, there are transitions from synchronized periodic motion to desynchronized quasiperiodicity and then to a fixed point. We find that the synchronous periodic and chaotic regimes are suppressed as a result of the coupling, while asynchronous quasiperiodic motion (the so-called anomalous regime) and the discrete analogue of amplitude death, namely the fixed point solution occupy large regions in parameter space. The present results differ from (and extend) an earlier study of time-delay coupled maps [18] where the transition from multistability to synchrony was described.

We have considered the case of 1-step delay which increases the dimension of the system by 1; generalization to the case of k-step delay is straightforward and the results (not presented here) generalize directly

$$x_{n+1} = (1-\beta) f(x_n) + \beta g(y_n),$$
$$y_{n+1} = (1-\beta) f(y_n) + \beta g(x_{n-k}). \tag{4}$$

We find that in general, the synchronization region reduces with the increase of delay, k. There is in-phase synchronization and a transition to a fixed point for odd k, while for even k there is anti-phase synchronization.

There is considerable current interest in the study of time-delayed systems since a finite velocity of information transmission is a feature of most natural systems. Interactions in many biological and physical systems can be characterized by time-delayed coupling [12]. The nature of synchronization in time-delay coupled systems with external forcing is an interesting extension of the above work, and is currently being investigated.

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