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Bifurcations and transitions in the quasiperiodically driven logistic map

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Abstract

We discuss several bifurcation phenomena that occur in the quasiperiodically driven logistic map. This system can have strange nonchaotic attractors (SNAs) in addition to chaotic and regular attractors; on SNAs the dynamics is aperiodic, but the largest Lyapunov exponent is nonpositive. There are a number of different transitions that occur here, from periodic attractors to SNAs, from SNAs to chaotic attractors, etc. We describe some of these transitions by examining the behavior of the largest Lyapunov exponent, distributions of finite time Lyapunov exponents and the invariant densities in the phase space. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Quasiperiodically driven logistic map; Strange nonchaotic attractors; Lyapunov exponent

1. Introduction

Quasiperiodically driven mappings where the dynamics can be strange (namely on fractal attractors) but non-chaotic (namely having a lack of sensitivity to initial conditions) are of considerable current interest [1,2]. Strange nonchaotic attractors (SNAs) were first described by Grebogi et al. in 1984 [3], and are now known to be generic in quasiperiodically driven systems. Such dynamics is paradoxical in some ways. The motion on SNAs is aperiodic, but over long times, nearby trajectories will coincide. The dynamics is almost always characterized by intermittency, which is indicative of the fact that such attractors are highly nonuniform and have a complicated interweaving of (locally) stable and unstable regions.

External forcing allows for additional means of probing nonlinear dynamical systems. If the forcing is periodic, then the motion of the system becomes either periodic or chaotic, but for quasiperiodic forcing (for example when a system is modulated with two frequencies which are incommensurate with each other) SNAs become possible. Strange nonchaotic dynamics usually occurs in the vicinity of strange chaotic behavior and periodic or quasiperiodic (nonstrange, nonchaotic) behavior. As a result, such systems can show transitions between dynamical states and bifurcation phenomena which are similar to those in analogous autonomous systems (for example the several scenarios [4] such as the period-doubling route to chaos, intermittency, attractor crises, etc.) as those that are distinct from the bifurcations of unforced systems [5–7]. These have been extensively studied in a number of

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different contexts: questions of interest range from the mechanisms through which SNAs are created [8–16,35–37], how they may be characterized [17,18], experimental systems where these might occur [18–20,38–40], etc.

In this paper we examine the transitions between a number of different types of attractors in the quasiperiodically driven logistic map. In this prototypical driven system, the attractors can be strange and nonchaotic, in addition to being strange and chaotic or nonchaotic and regular (torus attractors). Studies of the driven logistic map have played an important role in the study of SNAs. Kaneko [12] first observed "torus wrinkling" in this system: this was eventually described as the fractalization route to SNAs [13]. The creation of SNAs through the collision of stable period-doubled tori with their unstable parent torus was also first studied in this system [11]. One great advantage in studying the driven logistic mapping (even though it is somewhat difficult to obtain analytic results) is that the undriven logistic map has been extensively studied over the past few decades. Many of the features of the driven system find their parallels in the undriven system. At the same time, however, since the dynamics in the logistic map is generic of a wide class, the behavior that can be simply studied in the driven logistic map is a characteristic of most driven nonlinear systems. Some scenarios through which SNAs are formed in this and related systems have been reviewed recently [1].

SNAs occur in several different parameter ranges, between regions of periodic or torus attractors and regions of chaotic attractors. There is a plethora of possible dynamical transitions, some of which have parallels in the undriven system, such as torus bifurcations,

$$\dots nT \leftrightarrow 2nT \leftrightarrow \dots,$$

and others which do not, such as transitions from tori to SNAs,

$$\dots nT \leftrightarrow n \text{ band SNAs} \leftrightarrow \dots, \qquad \dots 2^n T \leftrightarrow 2^{n-1} \text{ band SNAs} \leftrightarrow \dots.$$

SNAs merge in a manner similar to the case of reverse bifurcations, widen in a manner similar to widening chaotic crises, and can transform from one type to another or from SNAs to chaotic attractors.

The main focus of this paper, in addition to characterizing the above bifurcations and transitions in this system through the Lyapunov exponent and its fluctuations, is also to examine the manner in which the invariant measure varies with the parameters of the system, and the effect that this has on the dynamics. In many instances, these bifurcations or transitions involve the Lyapunov exponent going through zero, and can be analyzed in terms of symmetries in the tangent-space dynamics.

In Section 2, we introduce the model, and discuss the analysis in terms of global and local Lyapunov exponents, and the return map for stretch exponents. The characteristic behavior of the Lyapunov exponents at the various transitions to different attractors is discussed in Section 3. This is followed by a summary in Section 4.

2. Model

We study the logistic map with quasiperiodic modulation of the parameter α,

$$x_{n+1} = \alpha[1 + \epsilon \cos 2\pi\theta_n]x_n(1 - x_n), \tag{1}$$

$$\theta_{n+1} = \theta_n + \omega \bmod 1, \tag{2}$$

where ω is taken to be an irrational number (usually the golden mean ratio, $(\sqrt{5} - 1)/2$). Successive iterates of θ will densely and uniformly cover the unit interval in a quasiperiodic manner, and the system therefore has *no* periodic orbits. As in our previous studies [5,6], we rescale the parameter

$$\epsilon' = \frac{\epsilon}{4/\alpha - 1}$$

for convenience, and study the system in the range $2 \le \alpha \le 4$ and $0 \le \epsilon' \le 1$.

The Lyapunov exponent corresponding to the θ-rotation is trivially zero; however, the other exponent, which is of most importance in determining the dynamics, varies with the parameters. It can be calculated by averaging the *stretch exponents* tangential to the flow, namely

$$y_n = \ln|\alpha[1 + \epsilon \cos 2\pi\theta_n](1 - 2x_n)|, \tag{3}$$

which is the local derivative of the mapping along a trajectory. The local or *N*-step Lyapunov exponent is

$$\lambda_N = \frac{1}{N}\sum_{j=1}^{N} y_j, \tag{4}$$

from which, asymptotically, one gets the global Lyapunov exponent

$$\Lambda = \lim_{N\to\infty} \lambda_N. \tag{5}$$

The stretch exponents are negative in the region bounded by the curves

$$x_\pm(\theta) = \frac{1}{2}\left[1 \pm \frac{1}{\alpha(1 + \epsilon \cos 2\pi\theta)}\right]. \tag{6}$$

Any change in the dynamics such that the invariant measure is increased in the region $[x_-(\theta), x_+(\theta)]$ will therefore lead to a decrease in the Lyapunov exponent, and conversely, depletion of measure in this region will naturally lead to an increase in the Lyapunov exponent.

Note, however, that the invariant measure, $\rho_{\alpha,\epsilon}(x, \theta)$, for this mapping is not known exactly (except at $\alpha = 4$, $\epsilon = 0$). It is therefore determined numerically by partitioning the phase space (x, θ) into bins and examining the itinerary of a long trajectory. Shown in Fig. 1(a) is an example of an SNA; the corresponding invariant measure is shown in Fig. 1(b). The solid line is the locus of $x_\pm(\theta)$ (cf. Eq. (6)), showing that the SNA is not only largely located within the contracting regions in phase space, but also has considerable support in the unstable regions.

Local LEs, λ_N, depend on initial conditions but (with probability 1) Λ does not. [1] In order to characterize the nonuniformity of the attractor, it has proved instructive to examine the distribution of local Lyapunov exponents. The probability density

$$P(N, \lambda)\,d\lambda = \text{probability that } \lambda_N \text{ lies between } \lambda \text{ and } \lambda + d\lambda \tag{7}$$

has been seen to have characteristic limiting forms that depend on the nature of the attractor [21,22]. Of course, as $N \to \infty$, $P(N, \lambda) \to \delta(\lambda - \Lambda)$, but the nature of the finite-size corrections and the approach to the limit are distinctive for different dynamical states. This analysis is relevant for the study of nonuniform attractors, particularly for different attractors across crisis points or at intermittency, when stretched exponentials often occur [21].

3. Results and discussion

In this section we study the variation of the dynamics through several transitions in the system as the parameters α and ϵ' are varied. Phase diagrams for this system in different parameter ranges have been obtained in a number of previous studies [5,6,10,11,22,23], and it is known that there are two distinct regions of chaotic dynamics corresponding, respectively, to high driving (large ϵ') and large nonlinearity (large α). These are shown in Fig. 2 as

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[1] Except for systems which have multiple coexisting attractors, such as the double kicked-rotor, Duffing oscillator, etc.
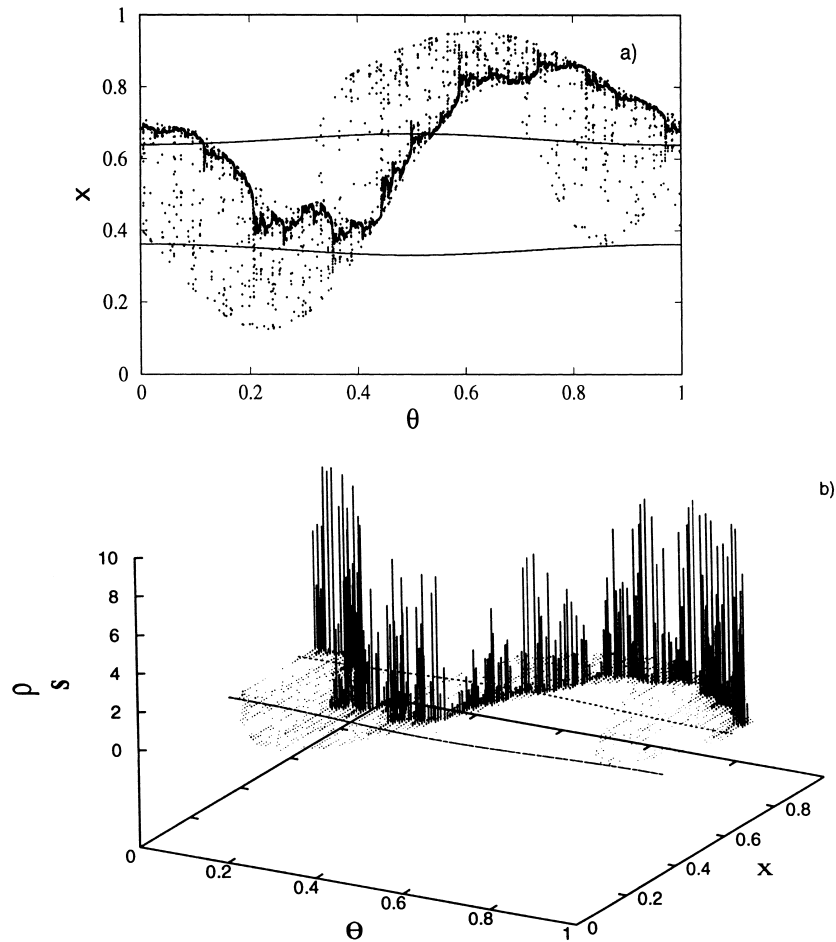
Fig. 1. (a) An SNA in the driven logistic map at $\epsilon' = 0.75$ and $\alpha = 3.26774$, and (b) the corresponding invariant density (scaled $\times 10^3$) in the phase space. The solid lines (in (a)) and the dashed lines (in (b)) are the loci of $x_{\pm}(\theta)$, Eq. (6).

C_2 and C_1 [5], and they separate a region of quasiperiodic (torus) dynamics. Strange nonchaotic motion occurs on the boundaries of these regions [6] as shown in Fig. 2.

In addition to the asymptotic nontrivial Lyapunov exponent Λ, we also examine the distribution of N-step Lyapunov exponents, and the variance of this distribution. The variation in Λ with α for fixed ϵ' is shown in Fig. 3(a). Although there are several bifurcations and transitions, these are not easily visible in the behavior of the Lyapunov exponent. We therefore examine an approximate or partial bifurcation diagram which can be obtained for this system by plotting the values of x that can be obtained within a narrow window in θ, namely in the interval $(\theta, \theta + \mathrm{d}\theta)$. This will depend upon the choice of $\mathrm{d}\theta$ and also on the particular value of θ chosen, but qualitative features of the bifurcation diagram are not affected.

The partial bifurcation diagram, Fig. 3(b), shows some of the transitions clearly: the torus-doubling, for instance, and the transition from torus attractors to fractal attractors. Since singularities are dense in θ, an SNA or a chaotic attractor shows up as a spread of points (see Fig. 3(b), for example), while a torus attractor appears as a point or a finite set of points. The distinction between a chaotic attractor or an SNA is not evident in the bifurcation diagram, but examining Fig. 3(a) and (b), and the change in the variance in Λ, shown in Fig. 3(c), gives a complete picture
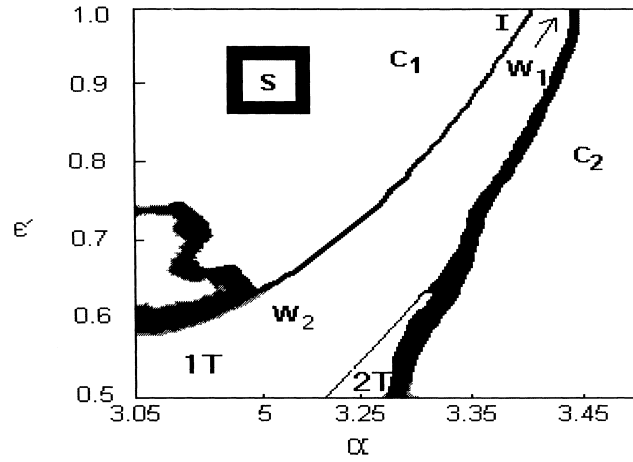
Fig. 2. Schematic phase diagram for a small region in parameter space for the forced logistic map (see details in [6]). T and C correspond to regions of torus and chaotic attractors. SNAs are mainly found in the shaded region along the boundary of T and C (marked S). W denotes the region where widening crises occur (see text).
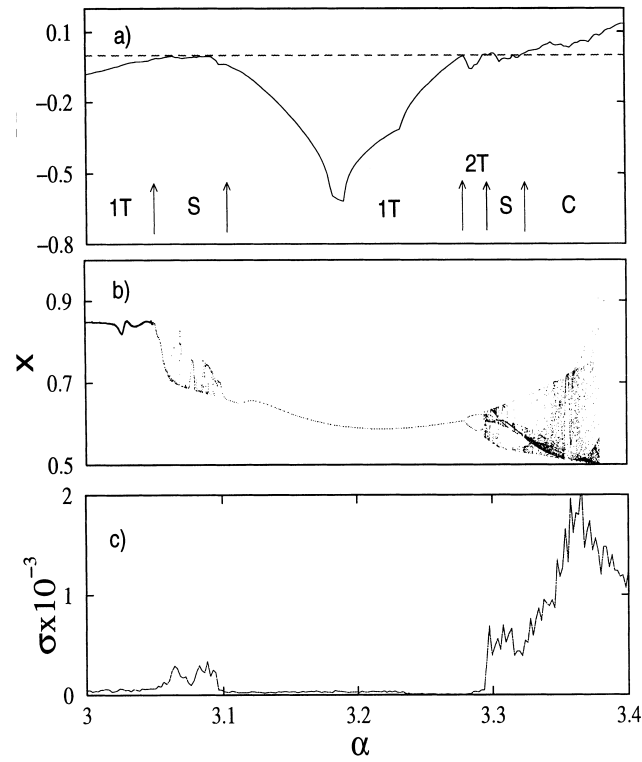

Fig. 3. (a) Variation of Λ as a function of α for fixed $\epsilon' = 0.595$. Note the highly oscillatory structure indicative of several transitions in the system. These are clearly shown in (b) which is a 'partial' bifurcation diagram with $\theta = 0.5$ and $d\theta = 0.0001$. The regions of torus attractors as well as strange behavior can be easily seen. (c) The variance in Λ. The symbols T and C correspond to torus and chaotic attractors, while S stands for SNA.

of the different attractors that are present in the system. (Our calculations of the variance, σ, are from 50 samples, each of total length 10^6 iterations.)

Quasiperiodic forcing converts the fixed points of the logistic map into tori. At the period-doubling bifurcation, the Lyapunov exponent is exactly zero, as in the unforced system, but with quasiperiodicity, the sequence of torus-doublings is interrupted and leads to SNAs, either through the collision of stable and unstable tori as discussed by Heagy and Hammel [11], or through fractalization [13,24]. This latter scenario is the most common route to SNA, and since there is, apparently, no bifurcation involved, it is not clearly understood as to how a torus gets increasingly wrinkled and transforms into a fractal attractor in the process. Other routes to SNA are known, some of which, like intermittency [5,25], occur in this system and others, such as the blowout bifurcation route [14], which do not.

3.1. From SNAs to SNAs: merging and widening crises

As has been remarked earlier, several of the bifurcation phenomena of (unforced) chaotic dynamical systems find their parallels in quasiperiodically forced systems. For example, there are analogs of crises phenomena. In the present system SNAs have a certain number of "bands". n-Band SNAs are formed from n-tori via fractalization, or from $2n$-tori through the Heagy–Hammel mechanism. Additionally, one-band SNAs can form either from a 1-torus [5] or a 3-torus [22,23] due to saddle-node bifurcations.

As parameters are further varied, SNAs themselves evolve and merge at quasiperiodic analogs of band-merging crises or reverse bifurcations: n-band SNAs transform to $n/2$-band SNAs. Through such a transition, when the dynamics remains nonchaotic and strange, Λ is a good order parameter. Sosnovtseva et al. [26], who discovered an example of this transition in the driven Hénon and circle maps, demonstrated the merging by examining the phase portrait. Given the fairly narrow range over which SNAs exist in any system, such transitions also occur in a restricted range, and often such crises can occur after the transition to chaotic attractors. However, the variation of Λ does not follow a uniform pattern as in the unforced case [27,28].

SNAs which are formed via fractalization may merge at negative Λ. The Lyapunov exponent *decreases* with increasing nonlinearity; shown in Fig. 4(a) is the variation of Λ at a band-merging bifurcation (shown by the arrow) from a two-band SNA to a one-band SNA for $\epsilon = 0.05$. The critical value of the parameter is $\alpha_c \approx 3.387439$ and the corresponding 'partial' bifurcation diagram of each alternate iterate of x is shown in Fig. 4(b). At merging crises in unforced systems, the Lyapunov exponent usually does not show an increase (unlike at widening crises), while with forcing, the exponent actually decreases.

This can be understood by examining the invariant density on the attractor before and after merging. The two bands of the two-band SNA straddle the region of phase space bounded by the curves $x_{\pm}(\theta)$, Eq. (6). After the merging transition, the density is enhanced mainly in this region where the stretch exponents are all negative. As a consequence, the Lyapunov exponent *must* decrease. (A plot of the difference in the invariant density on the SNAs before and after merging is shown in Fig. 4(c).) This appears to be the typical behavior at the SNA band-merging transition, [2] and is in contrast to the behavior of the unforced system.

Widening crises also occur, where the SNA abruptly changes size as a parameter is varied (near W_1 in Fig. 2). Shown in Fig. 5 is such an example, which occurs on the line $\epsilon' = 1$. Here, the post-crisis SNA, shown in Fig. 5(b), is clearly larger than the pre-crisis SNA (Fig. 5(a)) which is created via fractalization. The signatures of the transition are evident in both Λ and the variance (Fig. 5(c) and (d)). The sudden expansion of the attractor seems to be due to collision with the unstable saddle (which also gives rise to the intermittency transition to SNA [5]). This saddle

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[2] We have studied the forced ring map and Henon maps and find that at SNA mergings in these systems, the largest Lyapunov exponent decreases in a manner very similar to that in the logistic system.
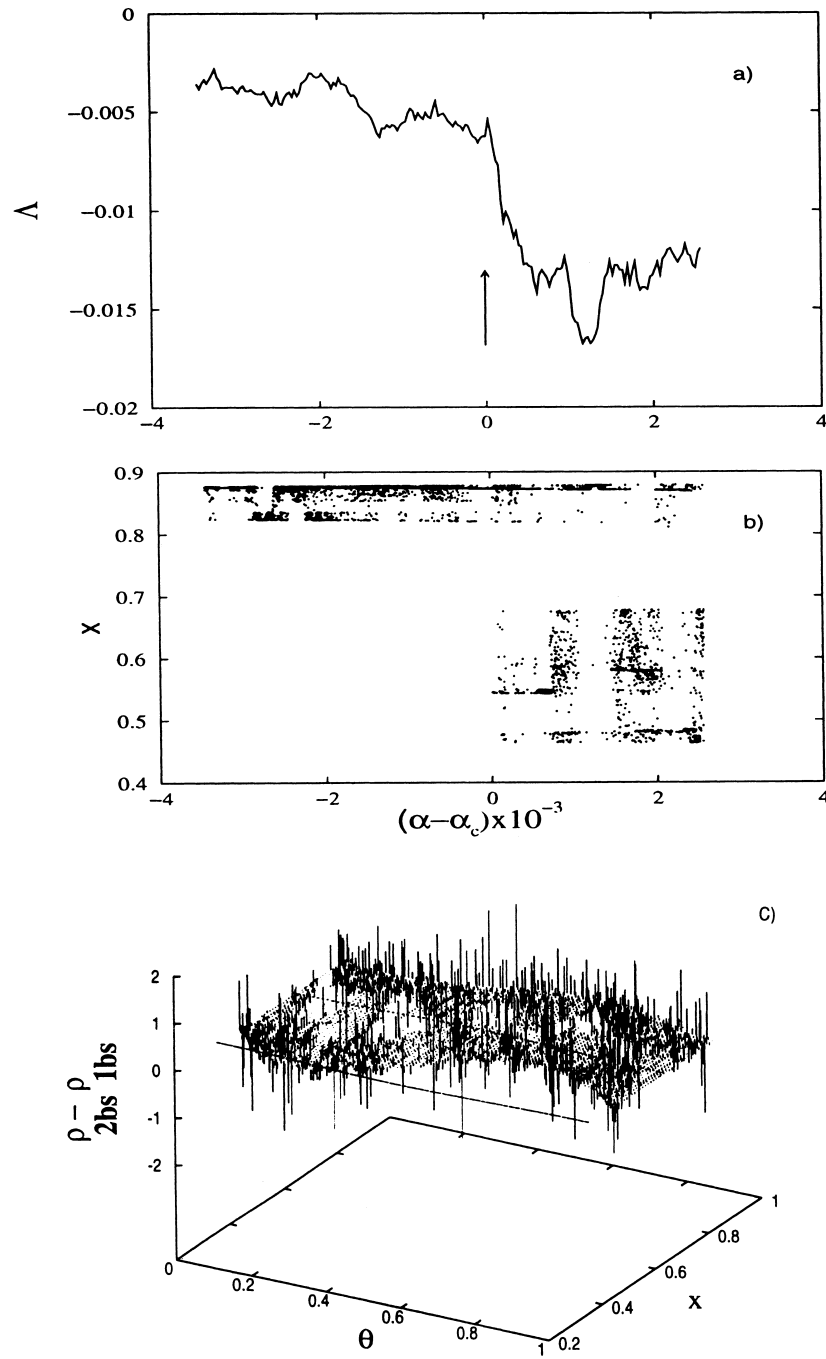
Fig. 4. (a) Variation of Λ at a band-merging bifurcation (shown by the arrow) from a two-band (fractalized) SNA to a one-band (Heagy–Hammel) SNA along $\epsilon = 0.05$ at $\alpha_c \approx 3.387439$, (b) the corresponding 'partial' bifurcation diagram of the second iteration for x in the interval $(\theta, \theta \pm d\theta)$, $\theta = 0.3$ and $d\theta = 0.01$, and (c) the difference in densities (scaled $\times 10^3$) after merging ($\alpha = 3.389$) and SNA before merging ($\epsilon = 3.386$) along $\epsilon = 0.05$. The subscripts 2bs and 1bs refer, respectively, to two-band and one-band SNAs.
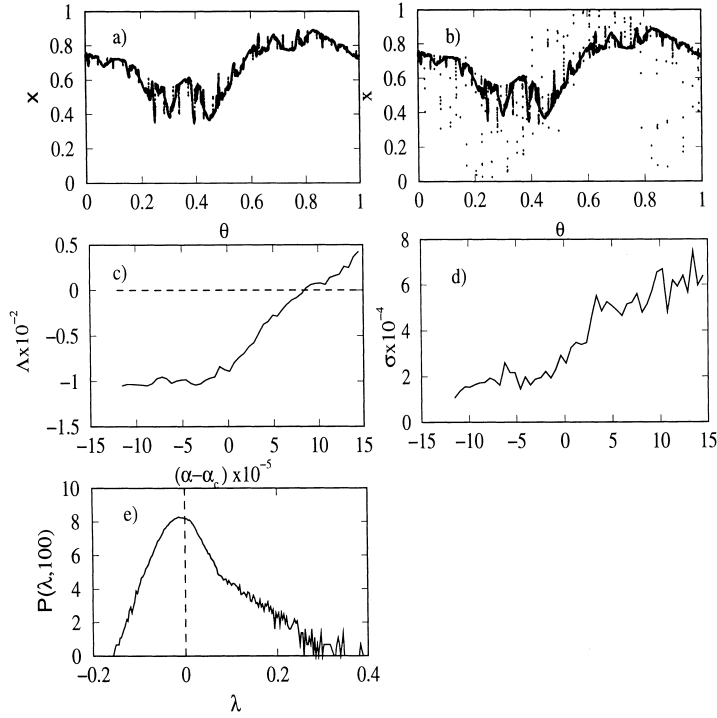
Fig. 5. (a) The SNA before widening along $\epsilon'=1$ at $\alpha = 3.4449$; (b) the SNA after the widening crisis at $\alpha = 3.44498$; (c) the variation of Lyapunov exponent, Λ, across the transition point $\alpha_c = 3.444955\ldots$; (d) the variance in Λ; (e) the distribution of finite-time Lyapunov exponents at $\alpha = 3.44498$, showing the stretched exponential tail which is characteristic of intermittency.

is difficult to locate, but in the three-dimensional extension of this system, namely the quasiperiodically driven Hénon map, Osinga and Feudel have recently described similar crises in detail [29] and have obtained the saddle there as well. In contrast to merging crises, the density is now enhanced in the unstable regions of phase space: Λ therefore increases. The distribution of finite-time LEs also has a stretched exponential tail which is characteristic of the crisis-induced intermittency [21] (see Fig. 5(e)). Similar interior crises also take place for lower forcing when a fractalized SNA transforms to an intermittent SNA. An example of this is shown in Fig. 6, near $\epsilon' = 0.614\ldots$ and $\alpha_c = 3.13188\ldots$ (W_2 in Fig. 2), namely on the edge of region C_2.

3.2. From SNAs to chaos

On variation of a parameter, an SNA can be transformed into a chaotic attractor if the largest Lyapunov exponent becomes positive. This transition, which is also not associated with any bifurcation, is not as dramatic as from the torus to an SNA. Typically there is a smooth variation in the Lyapunov exponent.

At the SNA $\rightarrow$ chaos transition, all the Lyapunov exponents of the system are zero. Since Λ (Eq. (5)) is a global average, it is of interest to know how the individual terms in Eq. (3) cancel out so as to give a value zero. Consider the partial finite sums [30],

$$\lambda_N^+ = \frac{1}{N_+}\sum_i y_i, \quad y_i > 0, \tag{8}$$

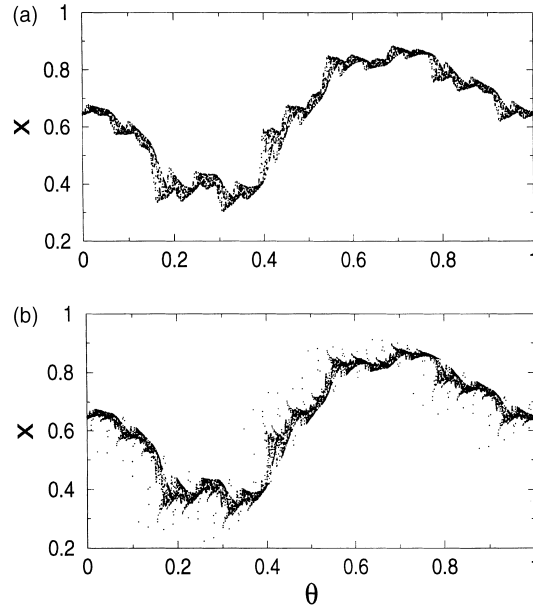$$\lambda_N^- = \frac{1}{N_-}\sum_i y_i, \quad y_i < 0, \tag{9}$$

Fig. 6. (a) The fractalized SNA (before widening) along $\epsilon' = 0.614$ at $\alpha = 3.13192$, and (b) the intermittent SNA after the widening crisis at $\alpha = 3.13184$.

namely the separate contributions to the local Lyapunov exponent. These are obtained by partitioning a trajectory into N_+ points on expanding regions where the stretch exponents are positive, and N_- points on contracting regions where the stretch exponents are negative with $N = N_+ + N_-$. Clearly $\lambda_N = (N_+/N)\lambda^+ + (N_-/N)\lambda^-$, and with the limits

$$\lim_{N \to \infty} \lambda_N^{\pm} \to \lambda^{\pm} \tag{10}$$

(cf. Eqs. (4) and (5)), $\Lambda = \lambda^+ + \lambda^-$.

If there are symmetries in the system [31,32],[3] then λ^+ and λ^- can equal each other and thereby yield $\Lambda = 0$. In such situations, there is a term-by-term cancelation of positive and negative stretch exponents ([32], see Footnote 3). At torus bifurcations, for instance, or at the blowout bifurcation transition to SNAs [3,14], this situation applies. On the other hand, there can be an "accidental" equality, namely, since both λ^+ and λ^- will be functions of parameters α and ϵ, it may happen that they are both equal in magnitude for some choice of parameters, and thereby lead to a zero value for the Lyapunov exponent. At the SNA to chaos transition, the latter situation occurs. Lai [16,37] investigated the transition from SNAs to chaotic attractors and found that this transition occurs only when the contraction and expansion rates for infinitesimal vectors along a typical trajectory on the attractor (namely λ^+ and λ^-) become equal; the LE passes through zero *linearly* as in Fig. 7(a).

There are, however, subtler effects that become apparent when examining fluctuations in the local Lyapunov exponents. The variance of the distribution shows a small increase across the transition (Fig. 7(b)). There is an increase in the fluctuations in the transition from torus attractors to SNA as well (from essentially zero to some finite value). At the SNA to chaos transition, the scale of fluctuation doubles, and is therefore noticeable on this scale. The SNA $\to$ chaotic attractor transition is not accompanied by any major change in the form or shape of

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[3] Note that the symmetries can be hidden as in [31] or complicated as at tangent bifurcations.
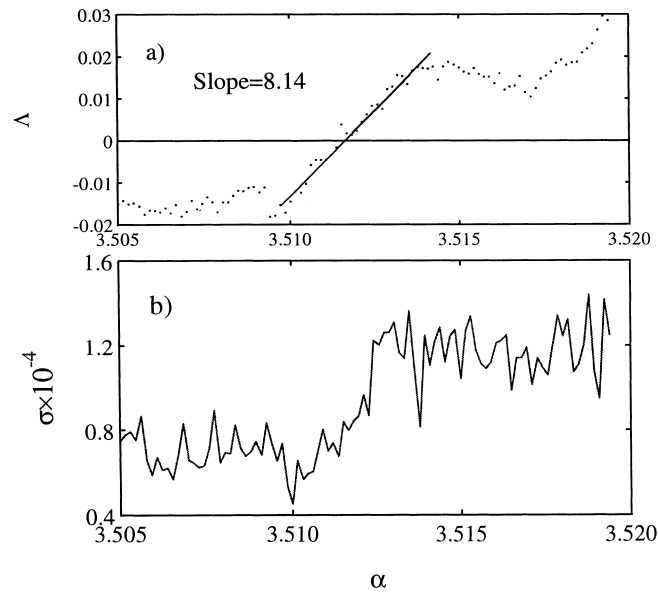
Fig. 7. The transition from SNA to a chaotic attractor along $\epsilon' = 0.3$. (a) The Lyapunov exponent across the transition, and (b) its fluctuations.

the attractor, and the Lyapunov exponent itself changes only from negative to positive. Yet the fluctuations on the chaotic attractor are always larger than those on the nonchaotic attractor.

This enhancement in fluctuations in Λ can be analyzed via the invariant density on the attractors. Shown in Fig. 8 is the *difference* in the invariant density on an SNA and a chaotic attractor symmetrically placed about the SNA $\rightarrow$ chaos transition. While it is clear that the morphology of the attractor does not change significantly, it can
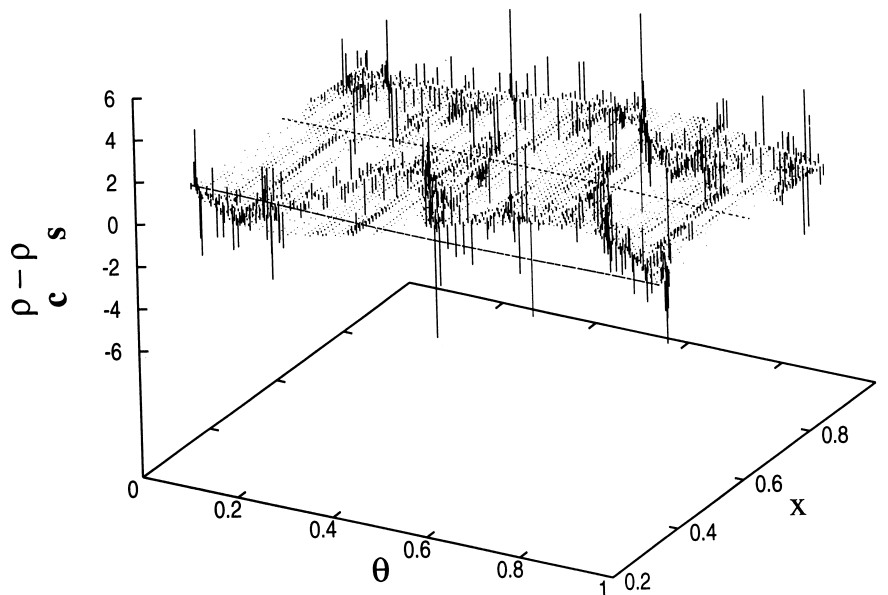

Fig. 8. The difference in densities (scaled $\times 10^3$) between a chaotic attractor ($\alpha = 3.515$) and SNA ($\alpha = 3.508$) along $\epsilon' = 0.3$. The plot of $x_\pm(\theta)$ (for $\alpha = 3.508$) of Eq. (6) is superimposed as a dashed line.

also be seen that on the SNA, the invariant density is enhanced in the regions where stretch exponents are mainly negative. Since these span the range from $-\infty$ to 0 within the region $[x_-(\theta), x_+(\theta)]$, they contribute significantly to the variance without greatly affecting the mean: the Lyapunov exponent thus does not change significantly, but the fluctuations show an increase.

Crises which occur after the transition to chaos appear to be very similar to analogous phenomena in systems without forcing [8,28]; Λ has a power-law dependence on the parameter, $\Lambda - \Lambda_c \propto (\alpha - \alpha_c)^\beta$. For example the widening crisis due to saddle-node bifurcation along $\epsilon' = 0.02$ across the transition point $\alpha_c = 3.85226\ldots$ is observed where $\beta \approx 0.67$ which is larger than in the unforced case [28].

4. Summary

In this paper we have studied different dynamical transitions that occur in the quasiperiodically driven logistic map, the main focus being on the quasiperiodic analogs of crises phenomena.

SNAs are formed via several different routes, and coexist with chaotic as well as other nonchaotic attractors. As a result, there are interesting transformations of one type of attractor to another. The torus to SNA transitions have been extensively described previously [1,6]. Fractalization, which is the most common scenario for the formation of SNAs is a "P2C2E": [4] no explicit bifurcation mechanism has been identified with this route. However, fractalized SNAs can be transformed into intermittent SNAs upon collision with an unstable torus in an analog of the interior crisis, and also undergo merging at the analog of a merging crisis.

In all these transitions, the nontrivial Lyapunov exponent is a good order-parameter, and furthermore, its fluctuations, the distributions of finite-time exponents, provide additional signatures for these transitions. In order to understand the nature of the variations of these quantities, however, it is necessary to study the invariant measure on SNAs. We show that they have support even in regions which are locally unstable.

At many of the transitions that involve such attractors, there is no particular change in form or morphology. However, by examining differences in the invariant density at different parameter values, it becomes clear that at these transitions, the manner in which the dynamics explores the attractor can change drastically, and small changes in the density can lead to significant effects in the fluctuation properties of such attractors.

Typical attractors in dynamical systems are nonuniform, both with respect to the natural invariant measure as well as in the rate at which nearby trajectories — locally — diverge or converge. If the measure on converging regions exceeds that on the diverging regions, then the attractor is asymptotically nonchaotic and will be characterized by a negative Lyapunov exponent. External modulation can be one method of altering the invariant measure, and this method of creating nonchaotic dynamics [30,34] therefore provides a new means of synchronization and control. In this context, therefore, studies of the bifurcations and transformations of SNAs give further insights into the interplay between global stability and local instability.

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